

# ON SINGULAR CALOGERO-MOSER SPACES

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**ABSTRACT.** Using combinatorial properties of complex reflection groups we show that if the group  $W$  is different from the wreath product  $\mathfrak{S}_n \wr \mathbb{Z}/m\mathbb{Z}$  and the binary tetrahedral group (labelled  $G(m, 1, n)$  and  $G_4$  respectively in the Shephard-Todd classification), then the generalised Calogero-Moser space  $X_{\mathbf{c}}$  associated to the centre of the rational Cherednik algebra  $H_{0,\mathbf{c}}(W)$  is singular for all values of the parameter  $\mathbf{c}$ . This result and a theorem of Ginzburg and Kaledin imply that there does not exist a symplectic resolution of the singular symplectic variety  $\mathfrak{h} \times \mathfrak{h}^*/W$  when  $W$  is a complex reflection group different from  $\mathfrak{S}_n \wr \mathbb{Z}/m\mathbb{Z}$  and the binary tetrahedral group (where  $\mathfrak{h}$  is the reflection representation associated to  $W$ ). Conversely it has been shown by Etingof and Ginzburg that  $X_{\mathbf{c}}$  is smooth for generic values of  $\mathbf{c}$  when  $W \cong \mathfrak{S}_n \wr \mathbb{Z}/m\mathbb{Z}$ . We show that this is also the case when  $W$  is the binary tetrahedral group. A theorem of Namikawa then implies the existence of a symplectic resolution in this case, completing the classification. Finally, we note that the above results together with work of Chlouveraki are consistent with a conjecture of Gordon and Martino on block partitions in the restricted rational Cherednik algebra.

## 1. INTRODUCTION

Let  $W$  be an irreducible complex reflection group and  $\mathfrak{h}$  its reflection representation. Etingof and Ginzburg [EG] associated to  $W$  a family of algebras, the *rational Cherednik algebras*  $H_{t,\mathbf{c}}(W)$ , depending on parameters  $t$  and  $\mathbf{c}$ . The definition is given in Section 2. When  $t = 0$ , these algebras have large centres and the geometry of the centre strongly influences the representation theory of the algebra. The affine variety  $X_{\mathbf{c}}$  corresponding to the centre of the rational Cherednik algebra was called the generalised Calogero-Moser space at  $\mathbf{c}$  by Etingof and Ginzburg. They showed [EG, Corollary 1.14], that for generic values of the parameter  $\mathbf{c}$ ,  $X_{\mathbf{c}}$  is smooth when  $W \cong G(m, 1, n)$ . However, Gordon [Go, Proposition 7.3] showed that, for many Weyl groups  $W$  not of type  $A$  or  $B(= C)$ ,  $X_{\mathbf{c}}$  is a singular variety for all choices of the parameter  $\mathbf{c}$ . Using similar methods we extend this result to all irreducible complex reflection groups.

**Theorem 1.1.** *Let  $W$  be an irreducible complex reflection group, not isomorphic to  $G(m, 1, n)$  or  $G_4$ , and  $X_{\mathbf{c}}$  the generalised Calogero-Moser space associated to  $W$ . Then  $X_{\mathbf{c}}$  is a singular variety for all choices of the parameter  $\mathbf{c}$ . Conversely for  $W \cong G_4$ ,  $X_{\mathbf{c}}$  is a smooth variety for generic values of  $\mathbf{c}$ .*

This completes the classification of rational Cherednik algebras for which  $X_{\mathbf{c}}$  is smooth for generic  $\mathbf{c}$ .

In [GK, Corollary 1.21], Ginzburg and Kaledin show that the existence of a symplectic resolution of the symplectic singularity  $\mathfrak{h} \times \mathfrak{h}^*/W$  implies that  $X_{\mathbf{c}}$  is smooth for generic  $\mathbf{c}$ . This result, together with Theorem 1.1 above implies the following corollary.

**Corollary 1.2.** *Let  $W$  be an irreducible complex reflection group with reflection representation  $\mathfrak{h}$ . Then there does not exist a symplectic resolution of  $\mathfrak{h} \times \mathfrak{h}^*/W$  when  $W \not\cong G(m, 1, n)$  or  $G_4$ .*

Since  $X_{\mathbf{c}}$  is smooth for generic values of  $\mathbf{c}$  when  $W \cong G_4$  and the symplectic singularity  $\mathfrak{h} \times \mathfrak{h}^*/G_4$  is four dimensional, a result of Namikawa [Na, Theorem 2.4] implies

**Corollary 1.3.** *There exists a symplectic resolution of the singular symplectic variety  $\mathfrak{h} \times \mathfrak{h}^*/G_4$ .*

In order to prove Theorem 1.1 we show that the restricted rational Cherednik algebra  $\bar{H}_{0,\mathbf{c}}(W)$  has irreducible representations of dimension  $< |W|$  for all values of  $\mathbf{c}$  when  $W$  is different from  $G(m, 1, n)$  and  $G_4$ . This implies that there exist blocks in  $\bar{H}_{0,\mathbf{c}}(W)$  with nonisomorphic irreducible modules. Therefore the corresponding block partition of  $\text{Irr}(W)$ , as described in [GM], is trivial for generic values of  $\mathbf{c}$  if and only if  $W$  is  $G(m, 1, n)$  or  $G_4$ . A conjecture of Gordon and Martino [GM] then implies that the partitioning of  $\text{Irr}(W)$  induced by the Rouquier families of the Hecke algebra  $\mathcal{H}_{\mathbf{q}}(W)$  should also be trivial for generic choices of  $\mathbf{c}$  if and only if  $W$  is  $G(m, 1, n)$  or  $G_4$ . Work of Chlouveraki [Ch] on the cyclotomic Hecke algebras of exceptional complex reflection groups shows that this is indeed the case.

## 2. THE RATIONAL CHEREDNIK ALGEBRA AT $t = 0$

**2.1. Definitions and notation.** Let  $W$  be a complex reflection group,  $\mathfrak{h}$  its reflection representation over  $\mathbb{C}$  with  $\dim(\mathfrak{h}) = n$ , and  $\mathcal{S}$  the set of all complex reflections in  $W$ . Let  $\omega : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow \mathbb{C}$  be the symplectic form on  $\mathfrak{h} \oplus \mathfrak{h}^*$  given by  $\omega((f_1, f_2), (g_1, g_2)) = f_2(g_1) - g_2(f_1)$  and  $\mathbf{c} : \mathcal{S} \rightarrow \mathbb{C}$  a  $W$ -invariant function. For  $s \in \mathcal{S}$ , define  $\omega_s : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow \mathbb{C}$  to be the restriction of  $\omega$  on  $\text{Im}(1 - s)$  and the zero form on  $\text{Ker}(1 - s)$ . The *rational Cherednik algebra* at parameter  $t = 0$ , as introduced by Etingof and Ginzburg [EG, page 250], is the quotient of the skew group algebra of the tensor algebra  $T(\mathfrak{h} \oplus \mathfrak{h}^*)$  with  $W$ ,  $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$ , by the ideal generated by the relations

$$(1) \quad [x, y] = \sum_{s \in \mathcal{S}} \mathbf{c}(s) \omega_s(x, y) s \quad \forall x, y \in \mathfrak{h} \oplus \mathfrak{h}^*$$

Let  $Z_{\mathbf{c}}$  denote the centre of  $H_{0,\mathbf{c}}$  and  $X_{\mathbf{c}} = \text{maxspec}(Z_{\mathbf{c}})$  the affine variety corresponding to  $Z_{\mathbf{c}}$ . The space  $X_{\mathbf{c}}$  is called the *generalised Calogero-Moser space* associated to the complex reflection group  $W$  at parameter  $\mathbf{c}$ . By [EG, Proposition 4.5], we have an inclusion  $A = \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \subset Z_{\mathbf{c}}$  and correspondingly a surjective morphism  $\Upsilon_{\mathbf{c}} : Z_{\mathbf{c}} \rightarrow \mathfrak{h}/W \times \mathfrak{h}^*/W$ . This allows us to define the restricted rational Cherednik algebra  $\bar{H}_{0,\mathbf{c}}(W)$  as

$$\bar{H}_{0,\mathbf{c}}(W) = \frac{H_{0,\mathbf{c}}(W)}{\langle A_+ \rangle}$$

where  $A_+$  denotes the ideal in  $A$  of elements with zero constant term. From the defining relations (1) we see that putting  $\mathfrak{h}^*$  in degree 1,  $\mathfrak{h}$  in degree  $-1$  and  $\mathbb{C}W$  in degree 0 defines a  $\mathbb{Z}$ -grading on the rational Cherednik algebra  $H_{t,\mathbf{c}}(W)$ . The ideal  $\langle A_+ \rangle$  is generated by elements that are homogeneous with respect to this grading, therefore  $\bar{H}_{0,\mathbf{c}}$  is also a  $\mathbb{Z}$ -graded algebra.

We denote by  $\mathbb{C}[\mathfrak{h}]^{coW}$  the coinvariant ring  $\mathbb{C}[\mathfrak{h}]/C[\mathfrak{h}]_+^W$ , where  $\mathbb{C}[\mathfrak{h}]_+^W$  is the ideal in  $\mathbb{C}[\mathfrak{h}]$  generated by the elements in  $\mathbb{C}[\mathfrak{h}]^W$  with zero constant term. We follow the notation introduced in [Go] and define

$$M(\lambda) := \bar{H}_{0,\mathbf{c}} \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \rtimes W} \lambda$$

to be the baby Verma  $\bar{H}_{0,\mathbf{c}}$ -module associated to the irreducible  $W$ -module  $\lambda$ . This module is a graded  $\bar{H}_{0,\mathbf{c}}$ -module with  $M(\lambda)_i = 0$  for  $i < 0$ . By [Go, Proposition 4.3],  $M(\lambda)$  has a simple head which we denote  $L(\lambda)$ .

We follow the notation of [Ste] with regards to complex reflection groups, and set  $d = m/p$  when considering the group  $G(m, p, n)$ . For an arbitrary  $\mathbb{Z}$ -graded vector space  $M = \oplus_{i \in \mathbb{Z}} M_i$ , the Poincaré polynomial of  $M$  will be denoted  $P(M, t)$ . We denote by  $f_\lambda(t)$  the *fake polynomial* of the irreducible representation  $\lambda$  of  $W$ . This is defined as

$$f_\lambda(t) := \sum_{i \in \mathbb{Z}_{\geq 0}} (\mathbb{C}[\mathfrak{h}]_i^{coW} : \lambda) t^i$$

where  $(\mathbb{C}[\mathfrak{h}]_i^{coW} : \lambda)$  is the multiplicity of  $\lambda$  in  $i^{th}$  degree of the coinvariant ring  $\mathbb{C}[\mathfrak{h}]^{coW}$  (thought of here as a graded  $W$ -module).

Let  $Irr(W)$  be a complete set of non-isomorphic irreducible representation of  $W$ . We will also require the surjective map  $\Theta : Irr(W) \rightarrow \Upsilon^{-1}(0)$ , taking  $\lambda$  to the annihilator of  $L(\lambda)$  in  $Z_{\mathbf{c}}$ , as defined in [Go, paragraph 5.4]. This map has the property that a fiber  $\Theta^{-1}(\mathfrak{m})$  is a singleton set if and only if  $\mathfrak{m}$  is a smooth closed point in  $X_{\mathbf{c}}$  ([Go, Theorem 5.6]).

**2.2. General results.** Let  $\{s_1, \dots, s_k\}$  be a conjugacy class consisting of complex reflections in  $W$  and  $\zeta$  the eigenvalue of  $s_1$  (and hence all  $s_i$ ) not equal to 1 when thinking of  $W$  as a subgroup of  $GL(\mathfrak{h})$ . For  $1 \leq i \leq k$ , let  $\omega_{s_i}$  be the restricted symplectic form on  $\mathfrak{h} \oplus \mathfrak{h}^*$  as defined above. Let  $\pi_{s_i} : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow \text{Im}(1 - s_i)$  be the projection map along  $\text{Ker}(1 - s_i)$ , so that  $\omega_{s_i} = \omega \circ \pi_{s_i}$ , and define  $\Omega = \sum_{i=1}^k \omega_{s_i}$ .

**Lemma 2.1.** *Let  $W$ ,  $\omega$  and  $\Omega$  be as above. Then  $\Omega = \frac{k}{n}(1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1}(2 - \zeta - \zeta^{-1})\omega$ .*

*Proof.* Since each  $\omega_{s_i}$  is alternating and  $\mathbb{C}$ -linear,  $\Omega \in \bigwedge^2(\mathfrak{h} \oplus \mathfrak{h}^*)$ . Let  $x \in \mathfrak{h} \oplus \mathfrak{h}^*$ . Then  $x$  decomposes uniquely as  $x_1 + x_2$ , with  $x_1 \in \text{Im}(1 - s_i)$  and  $x_2 \in \text{Ker}(1 - s_i)$ . By definition, there exists  $y \in \mathfrak{h} \oplus \mathfrak{h}^*$  such that  $(1 - s_i)y = x_1$ . Then  $(1 - gs_i g^{-1})(gy) = g(1 - s_i)g^{-1}gy = g(1 - s_i)y = gx_1$  implying that  $gx_1 \in \text{Im}(1 - gs_i g^{-1})$ . Also  $(1 - s_i)x_2 = 0$  implies that  $(1 - gs_i g^{-1})gx_2 = 0$  hence  $gx$  decomposes as  $gx_1 + gx_2$  with  $gx_1 \in \text{Im}(1 - gs_i g^{-1})$  and  $gx_2 \in \text{Ker}(1 - gs_i g^{-1})$ . Therefore  $\pi_{gs_i g^{-1}} = g\pi_{s_i}g^{-1}$  and  $\omega_{s_i}(g^{-1}x, g^{-1}y) = \omega_{gs_i g^{-1}}(x, y)$ . Hence  $\Omega \in (\bigwedge^2(\mathfrak{h}^* \oplus \mathfrak{h}))^W$ . By [EG, Lemma 2.23]  $\dim(\bigwedge^2(\mathfrak{h}^* \oplus \mathfrak{h}))^W = 1$ , therefore there exists  $\lambda \in \mathbb{C}$  such that  $\Omega = \lambda\omega$ . Consider  $\Omega'(x, y) = \Omega((x, 0), (0, y))$ , where  $x \in \mathfrak{h}$  and  $y \in \mathfrak{h}^*$ . Recall that  $\zeta$  is the eigenvalue of  $s_i$  not equal to 1, then  $\pi_{s_i}(x) = (1 - \zeta)^{-1}(1 - s_i)x$  and  $\pi_{s_i}(y) = (1 - \zeta^{-1})^{-1}(1 - s_i)y$ . Expanding  $\Omega'(x, y)$

$$\Omega'(x, y) = \sum_{i=1}^k \omega((1 - \zeta)^{-1}(1 - s_i)x, (1 - \zeta^{-1})^{-1}(1 - s_i)y)$$

$$\begin{aligned}
&= (1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1} \sum_{i=1}^k [\omega(x, y) - \omega(s_i x, y) - \omega(x, s_i y) + \omega(s_i x, s_i y)] \\
&= (1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1} \omega(x, (\sum_{i=1}^k 2 - s_i - s_i^{-1})y)
\end{aligned}$$

Define  $\phi = (\sum_{i=1}^k 2 - s_i - s_i^{-1}) : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ , a  $W$ -homomorphism. The trace of  $\phi$  is  $2nk - (n-1)k - k\zeta - (n-1)k - k\zeta^{-1} = k(2 - \zeta - \zeta^{-1})$ . Since  $\mathfrak{h}^*$  is irreducible, Schur's lemma says that  $\phi(y) = \frac{k}{n}(2 - \zeta - \zeta^{-1})y$  and therefore  $\lambda = \frac{k}{n}(1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1}(2 - \zeta - \zeta^{-1})$ .  $\square$

We also require the notion of a generalised baby Verma module, which are baby Verma modules above points other than the origin in  $\mathfrak{h}/W \times \mathfrak{h}^*/W$ .

**Definition 2.2.** Let  $(p, q) \in \mathfrak{h}/W \times \mathfrak{h}^*$ ,  $W_q$  the stabiliser subgroup of  $q$  in  $W$  and  $E$  an irreducible  $W_q$ -module. Then we define the *generalised baby Verma* module

$$\Delta_{\mathbf{c}}(E; p, q) := H_{0, \mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*] \rtimes W_q} E$$

where the action of  $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*] \rtimes W_q$  on  $E$  is given by  $(f \otimes g \otimes w) \cdot e = f(p)g(q)w \cdot e$  for all  $f \in \mathbb{C}[\mathfrak{h}]^W$ ,  $g \in \mathbb{C}[\mathfrak{h}^*]$ ,  $w \in W_q$ .

Since  $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \subseteq Z_{\mathbf{c}}$ , Schur's lemma implies that, for every irreducible  $H_{0, \mathbf{c}}$ -module  $L$ , there exists  $(p, r) \in \mathfrak{h}/W \times \mathfrak{h}^*/W$  such that  $(f \otimes g) \cdot l = f(p)g(r)l$ , for all  $l \in L$ ,  $f, g \in \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ . Choosing a point  $q$  in the orbit represented by  $r$  we write  $(p, r) = (p, Wq)$  and say that the irreducible  $H_{0, \mathbf{c}}$ -module  $L$  lies above  $(p, Wq)$ .

**Lemma 2.3.** Let  $L$  be an irreducible  $H_{0, \mathbf{c}}$ -module lying above  $(p, Wq)$ . Then there exist  $E \in \text{Irr}(W_q)$  and a surjective  $H_{0, \mathbf{c}}$ -homomorphism  $\phi : \Delta_{\mathbf{c}}(E; p, q) \twoheadrightarrow L$ .

*Proof.* The action on  $L$  of the commutative ring  $\mathbb{C}[\mathfrak{h}^*]$  gives a decomposition  $L = \bigoplus_{q' \in \mathfrak{h}^*} L_{q'}^{\text{gen}}$  of  $L$  into generalised eigenspaces. That is, for each  $l \in L_{q'}^{\text{gen}}$  and  $f \in \mathbb{C}[\mathfrak{h}^*]$ , there exists an  $N \in \mathbb{N}$  such that  $(f - f(q'))^N \cdot l = 0$  (since  $L$  is finite dimensional, we can choose  $N$  to be independent of  $f$  and  $l$ ).

Choose  $q'$  such that  $L_{q'}^{\text{gen}} \neq 0$ , so that  $(f - f(q'))^N$  acts as zero on  $L_{q'}^{\text{gen}}$  for all  $f \in \mathbb{C}[\mathfrak{h}^*]^W$ . As  $L$  lies over  $(p, Wq)$  we see that  $(f - f(q))$  also acts nilpotently on  $L_{q'}^{\text{gen}}$  and  $f(q) = f(q')$ . Since  $W$  is a finite group, each orbit in  $\mathfrak{h}^*$  is closed, therefore  $q' \in Wq$  and we can find  $w \in W$  such that  $w \cdot q = q'$ . Now let  $0 \neq L_{q'} \subseteq L_{q'}^{\text{gen}}$  be the space of elements  $l$  in  $L_{q'}^{\text{gen}}$  such that  $(f - f(q')) \cdot l = 0$ , for all  $f \in \mathbb{C}[\mathfrak{h}^*]$ . Then  $w^{-1}(L_{q'}) \neq 0$  and  $f \cdot (w^{-1}l) = (fw^{-1}) \cdot l = w^{-1} \cdot (w f \cdot l) = w^{-1} \cdot (w f)(q')l = w^{-1} \cdot (f(w^{-1}q'))l = f(q)w^{-1} \cdot l$  implies that  $w^{-1}(L_{q'}) \subseteq L_q$ . Thus  $L_q$  is a nonzero  $W_q$ -submodule of  $L$  because  $f \cdot (v \cdot l) = (fv) \cdot l = v \cdot (v^{-1}f) \cdot l = v \cdot f(vq)l = v \cdot f(q)l = f(q)(v \cdot l)$  for all  $f \in \mathbb{C}[\mathfrak{h}]$ ,  $v \in W_q$  and  $l \in L_q$ . Choose an irreducible  $W_q$ -submodule  $E$  of  $L_q$ . The inclusion  $E \hookrightarrow L$  induces a  $H_{0, \mathbf{c}}$ -homomorphism  $\phi : \Delta_{\mathbf{c}}(E; p, q) \rightarrow L$ . The fact that  $L$  is irreducible implies that this is a surjection.  $\square$

### 3. SINGULAR GENERALISED CALOGERO-MOSER SPACES

#### 3.1. The main result.

**Theorem 3.1.** *For all  $W$  not isomorphic to  $G(m, 1, n)$  or  $G_4$  and for all parameters  $\mathbf{c}$ , the variety  $X_{\mathbf{c}}$  is singular.*

By [EG, Proposition 3.8] the statement of Theorem 3.1 is equivalent to the statement: *for  $W$  not isomorphic to  $G(m, 1, n)$  or  $G_4$  and for all parameters  $\mathbf{c}$  there exists an irreducible  $H_{0,\mathbf{c}}(W)$ -module  $L$  with  $\dim L < |W|$ .* Therefore Theorem 3.1 follows from

**Proposition 3.2.** *For each  $W$  not isomorphic to  $G(m, 1, n)$  or  $G_4$ , there exists an irreducible  $W$ -module  $\lambda$  such that for all parameters  $\mathbf{c}$ , the irreducible  $\bar{H}_{0,\mathbf{c}}(W)$ -module  $L(\lambda)$  has dimension  $< |W|$ .*

The proof of Proposition 3.2 will occupy the remainder of Section 3. The irreducible complex reflection groups were classified by Shephard and Todd [ST] and either belong to an infinite family labelled  $G(m, p, n)$ , where  $m, p, n \in \mathbb{N}$  and  $p|m$ , or to one of 34 exceptional groups  $G_4, \dots, G_{37}$ .

**Lemma 3.3.** *Let  $W$  be a complex reflection group. Let  $\lambda \in \text{Irr}(W)$  be the unique representation corresponding to a smooth point of  $\Upsilon^{-1}(0)$  in  $X_{\mathbf{c}}$  i.e.  $\Theta(\lambda)$  is smooth in  $X_{\mathbf{c}}$ . Then the Poincaré polynomial of  $L(\lambda)$  as a graded vector space is given by*

$$(2) \quad P(L(\lambda), t) = \frac{\dim(\lambda) t^{b_{\lambda^*}} P(\mathbb{C}[\mathfrak{h}^*]^{coW}, t)}{f_{\lambda^*}(t)}$$

where  $\lambda^*$  is the irreducible  $W$ -module dual to  $\lambda$ , and  $b_{\lambda}$  the trailing degree of the fake polynomial  $f_{\lambda}(t)$ .

*Proof.* By [Go, Lemma 4.4, paragraphs (5.2) and (5.4)], the graded composition factors of  $M(\lambda)$  are all of the form  $L(\lambda)[i]$ , for some  $i \geq 0$ . Therefore we can find a multiset  $\{i_1, \dots, i_k\}$  such that as a graded  $W$ -module

$$M(\lambda) \cong L(\lambda)[i_1] \oplus L(\lambda)[i_2] \oplus \dots \oplus L(\lambda)[i_k].$$

Since  $\Theta(\lambda)$  is a smooth point in  $X_{\mathbf{c}}$ , [EG, Theorem 1.7] says that  $L(\lambda) \cong \mathbb{C}W$  as a  $W$ -module. Hence it contains a unique copy of the trivial representation  $T$ . Assume this copy occurs in degree  $a$  in  $L(\lambda)$ . Then it will occur in degree  $a - i_j$  in  $L(\lambda)[i_j]$ . As a graded  $W$ -module,  $M(\lambda) \cong \mathbb{C}[\mathfrak{h}^*]^{coW} \otimes \lambda$ . The fact that  $[\tau \otimes \lambda : T] = \delta_{\tau\lambda^*}$  implies that the graded multiplicity of  $T$  in  $M(\lambda)$  equals the graded multiplicity of  $\lambda^*$  in  $\mathbb{C}[\mathfrak{h}^*]^{coW}$ . The graded multiplicity of  $\lambda^*$  in  $\mathbb{C}[\mathfrak{h}^*]^{coW}$  is  $f_{\lambda^*}(t)$ . Hence  $P(M(\lambda), t) = t^{-a} f_{\lambda^*}(t) P(L(\lambda), t)$ . The lowest nonzero term of  $P(L(\lambda), t)$  occurs in degree zero implying that  $a = b_{\lambda^*}$ . The formula follows by noting that  $P(M(\lambda), t)$  is  $\dim(\lambda) P(\mathbb{C}[\mathfrak{h}^*]^{coW})$ .  $\square$

Since  $L(\lambda)$  is a finite dimensional module, the above lemma shows that the right hand side of equation (2) is a polynomial in  $\mathbb{Z}[t, t^{-1}]$  with integer coefficients. Moreover, [Go, Lemma 4.4] shows that it is actually in  $\mathbb{Z}[t]$  and that the degree 0 coefficient is 1.

**3.2. The infinite series.** We show that for  $p \neq 1$  and  $W = G(m, p, n) \neq G(2, 2, 3)$  we can choose an irreducible representation  $\lambda$  of  $G(m, p, n)$  such that Lemma 3.3 does not hold. Thus  $L(\lambda)$  will have dimension  $< |G(m, p, n)|$ , proving Proposition 3.2 in this case. The group  $G(2, 2, 3)$  is the Weyl group corresponding

to the Dynkin diagram  $D_3 = A_3$  and hence  $G(2, 2, 3) \cong S_4$ . By [EG, Corollary 16.2],  $X_{\mathbf{c}}$  is smooth for generic and hence all non-zero  $\mathbf{c}$  in this case.

We give a description of the parameterization of irreducible  $G(m, p, n)$ -modules. The reader should consult [Ste, pages 379-381] for details. An  $m$ -multipartition  $\underline{\lambda}$  of  $n$  is an ordered  $m$ -tuple of partitions  $(\lambda^0, \dots, \lambda^{m-1})$  such that  $|\lambda^0| + \dots + |\lambda^{m-1}| = n$ . Let  $\mathcal{P}(m)$  denote the set of all  $m$ -multipartitions of  $n$ . There is an action of the cyclic group  $\mathbb{Z}/p\mathbb{Z} = \langle \omega \rangle$  on  $\mathcal{P}(m)$  given by  $\omega \cdot (\lambda^0, \dots, \lambda^{m-1}) = (\lambda^{m+1-d}, \lambda^{m+2-d}, \dots, \lambda^{1+d}, \dots, \lambda^{m-d})$ , where the superscript is taken mod  $m$  (recall from Subsection 2.1 that  $d = m/p$ ). For  $\underline{\lambda} \in \mathcal{P}(m)$ , we denote the orbit  $\mathbb{Z}/p\mathbb{Z} \cdot \underline{\lambda}$  by  $\{\underline{\lambda}\}$  and  $\text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\underline{\lambda}) \leq \mathbb{Z}/p\mathbb{Z}$  is the stabiliser subgroup with respect to  $\underline{\lambda}$ . Then the irreducible representations of  $G(m, p, n)$  are labelled by distinct pairs  $(\{\underline{\lambda}\}, \epsilon)$ , where  $\epsilon \in \text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\underline{\lambda})$ .

Let  $(t)_{(n)} = (1-t) \dots (1-t^{n-1})(1-t^n)$  and for  $\lambda$  a partition of  $n$ , denote by  $n(\lambda) = \sum_{i=1}^k (i-1)\lambda_i$  the partition statistic. The young diagram  $D_\lambda$  of a partition  $\lambda$  is the finite subset of  $\mathbb{N} \times \mathbb{N}$ , justified to the south west (in the French style), representing  $\lambda$ . For  $(i, j) \in D_\lambda$ , we denote by  $h(i, j)$  the hook length at  $(i, j)$ . The hook polynomial is defined to be

$$H_\lambda(t) = \prod_{(i,j) \in D_\lambda} (1 - t^{h(i,j)})$$

[Ste, Corollary 6.4] states that the fake polynomial of the irreducible representation labelled by  $(\{\underline{\lambda}\}, \epsilon)$  is

$$(3) \quad f_{\{\underline{\lambda}\}}(t) = \frac{1 - t^{dn}}{1 - t^{mn}} R_{\{\underline{\lambda}\}}(t) I_{\underline{\lambda}}(t^m)$$

where

$$R_{\{\underline{\lambda}\}}(t) = \sum_{\underline{\mu} \in \{\underline{\lambda}\}} t^{r(\underline{\mu})} \quad \text{with} \quad r(\underline{\mu}) = \sum_{i=0}^{m-1} i|\mu^i| \quad \text{and} \quad I_{\underline{\lambda}}(t) = (t)_{(n)} \prod_{i=1}^m \frac{t^{n(\lambda^i)}}{H_{\lambda^i}(t)}$$

Note that the formula only depends on the orbit and not on the choice of stabiliser.

We wish to calculate the rational function (2) for a well chosen representation  $(\{\underline{\mu}\}, \epsilon)$  of the irreducible representations of  $G(m, p, n)$ . By [Hu, Theorem 3.15], the Poincaré polynomial of the coinvariant ring of  $W$  is given by

$$P(\mathbb{C}[\mathfrak{h}^*]^{coW}, t) = \prod_{i=1}^n \frac{1 - t^{d_i}}{1 - t}$$

where  $d_1, \dots, d_n$  are the degrees of a set of fundamental homogeneous invariant polynomials of  $W$  ( $d_1, \dots, d_n$  are independent, up to reordering, of the polynomials choosen). By [ST, page 291],  $d_1, \dots, d_n = m, 2m, \dots, (n-1)m, dn$  when  $W = G(m, p, n)$ .

Hence, if the dual representation of  $(\{\underline{\mu}\}, \epsilon)$  is  $(\{\underline{\lambda}\}, \eta)$ , equation (2) becomes

$$\begin{aligned}
(4) \quad & P(L(\{\underline{\mu}\}, \epsilon), t) = \\
& \frac{\dim(\{\underline{\mu}\}, \epsilon) t^{b_{\{\underline{\lambda}\}}} (1-t^m)(1-t^{2m}) \dots (1-t^{(n-1)m})(1-t^{nd}) \prod_{i=0}^{m-1} H_{\lambda^i}(t^m)(1-t^{mn})}{(1-t)^n (1-t^{dn}) R_{\{\underline{\lambda}\}}(t) (t^m)_{(n)} \prod_{i=0}^{m-1} t^{n(\lambda^i)m}} \\
& = \frac{\dim(\{\underline{\mu}\}, \epsilon) t^{b_{\{\underline{\lambda}\}}} \prod_{i=0}^{m-1} H_{\lambda^i}(t^m)}{(1-t)^n R_{\{\underline{\lambda}\}}(t) \prod_{i=0}^{m-1} t^{n(\lambda^i)m}}
\end{aligned}$$

Let  $k \in \mathbb{N}$  such that  $t^k \mid R_{\{\underline{\lambda}\}}(t)$  but  $t^{k+1} \nmid R_{\{\underline{\lambda}\}}(t)$  in  $\mathbb{Z}[t]$  and write  $R_{\{\underline{\lambda}\}}(t) = t^k \tilde{R}_{\{\underline{\lambda}\}}(t)$ . Then rearrange equation (3) as

$$(5) \quad f_{\{\underline{\lambda}\}}(t) = \left( t^k \prod_{i=0}^{m-1} t^{n(\lambda^i)m} \right) \tilde{R}_{\{\underline{\lambda}\}}(t) \left( \frac{1-t^{dn}}{1-t^{mn}} (t^m)_{(n)} \prod_{i=1}^m \frac{1}{H_{\lambda^i}(t^m)} \right)$$

Since each  $H_{\lambda^i}(t^m)$  is a product of factors of the form  $(1-t^l)$ , the product in the right most bracket consists entirely of factors of the form  $(1-t^l)$ . Therefore

$$t^{b_{\{\underline{\lambda}\}}} = t^k \prod_{i=0}^{m-1} t^{n(\lambda^i)m}$$

and equation (4) becomes

$$(6) \quad P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon) \prod_{i=1}^m H_{\lambda^i}(t^m)}{(1-t)^n \tilde{R}_{\{\underline{\lambda}\}}(t)}.$$

To contradict Lemma 3.3 and hence prove Proposition 3.2 we have

**Lemma 3.4.** *Let  $p \neq 1$  and  $W = G(m, p, n)$  with  $W \neq G(2, 2, 3)$ . Then there exists  $(\{\underline{\mu}\}, \epsilon) \in \text{Irr}(W)$  such that the right hand side of equation (6) is not an element of  $\mathbb{C}[t]$ .*

*Proof.* We consider the cases  $n = 2, 3$  and  $n > 3$  separately. For  $n > 3$  choose  $(\{\underline{\mu}\}, \epsilon)$  such that its dual representation is  $\underline{\lambda} = (\lambda^0, \emptyset, \dots, \emptyset)$ , where  $\lambda^0 = (2, 2, 1, 1, \dots, 1)$ . Then

$$\tilde{R}(t) = R(t) = 1 + t^{dn} + t^{2dn} + \dots + t^{(p-1)dn} = \frac{1-t^{p dn}}{1-t^{dn}}$$

and for this particular  $m$ -multipartition we have

$$\prod_i H_{\lambda^i}(t^m) = H_{\lambda^0}(t^m) = (1-t^{2m})(1-t^m)(1-t^{(n-1)m})(1-t^{(n-2)m}) \prod_{i=1}^{n-4} (1-t^{im})$$

Equation (6) becomes

$$(7) \quad P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon)(1 - t^{2m})(1 - t^m)(1 - t^{(n-1)m})(1 - t^{(n-2)m})(t^m)_{n-4}(1 - t^{dn})}{(1 - t^{mn})(1 - t)^n}.$$

The numerator of (7) factorises over  $\mathbb{C}$  as a product of factors  $(1 - \omega t)$ , where  $\omega$  is a primitive  $k^{th}$  root of unity with  $1 \leq k < mn$ , whereas the denominator contains at least one factor of the form  $(1 - \sigma t)$ , where  $\sigma$  is a primitive  $mn^{th}$  root of unity. Therefore, since  $\mathbb{C}[t]$  is an Euclidean domain, the right hand side of (7) cannot not lie in  $\mathbb{C}[t]$ .

For  $n = 2$  and  $m \geq n$ , take  $\underline{\Delta} = ((1), (1), \emptyset \dots \emptyset)$ . Then

$$\prod_i H_{\lambda^i}(t^m) = (1 - t^m)^2 \quad R(t) = \frac{t(1 - t^{2m})}{1 - t^{2d}} \text{ and } \tilde{R}(t) = \frac{1 - t^{2m}}{1 - t^{2d}}.$$

Substituting into (6)

$$P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon)(1 - t^m)^2(1 - t^{2d})}{(1 - t^{2m})(1 - t)^2}$$

By the same reasoning as above, since  $2m > 2d, m$ , this rational function is not a polynomial.

Similarly, for  $n = 3$  and  $m \geq n$ , take  $\underline{\Delta} = ((1), (1), (1), \emptyset \dots \emptyset)$ . Then

$$\prod_i H_{\lambda^i}(t^m) = (1 - t^m)^3 \quad R(t) = \frac{t^3(1 - t^{3m})}{1 - t^{3d}} \text{ and } \tilde{R}(t) = \frac{1 - t^{3m}}{1 - t^{3d}}.$$

Substituting into (6)

$$P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon)(1 - t^m)^3(1 - t^{3d})}{(1 - t^{3m})(1 - t)^3}.$$

Once again, this rational function is not a polynomial because  $3m > 3d, m$ . □

Therefore, for all  $W = G(m, p, n)$ ,  $p > 1$ , and with  $W \neq G(2, 2, 3)$ , we have found an irreducible representation  $(\{\underline{\mu}\}, \epsilon)$  of  $W$  such that the Poincaré polynomial of the corresponding irreducible  $\bar{H}_{0, \mathbf{c}}(W)$ -module  $L(\{\underline{\mu}\}, \epsilon)$  cannot be of the form given in Lemma 3.3. Hence the dimension of  $L(\{\underline{\mu}\}, \epsilon)$  must be less than  $|W|$ . Our argument is independent of the parameter  $\mathbf{c}$ , therefore we have proved Proposition 3.2 in this case.

**3.3. The Exceptional Groups.** Using the computer algebra program [GAP, GAP] together with the package [CHE, CHEVIE] we calculate for each exceptional complex reflection group  $W$  (excluding  $G_4$ ), the number of irreducible representations  $\lambda$  for which the polynomial  $t^{-b^*} f_{\lambda^*}(t)$  does not divide  $P(\mathbb{C}[\mathfrak{h}]^{coW}, t)$  in  $\mathbb{C}[t]$ . Table (3.3) gives the results of these calculations. For each of these  $\lambda$ , Lemma 3.3 does not hold and hence  $\dim L(\lambda) < |W|$  for all values of  $\mathbf{c}$ . Since this number is always positive, Proposition 3.2 is proved for the exceptional groups.



TABLE 1. Number of irreducibles that fail Lemma 3.3

Group	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
# failures	3	6	13	2	16	15	43	1	4	9	18	15	55	70	164	18	42	12	4
Group	24	25	26	27	28	29	30	31	32	33	34	35	36	37					
# failures	8	3	10	26	5	24	24	40	33	30	148	9	30	75					

The code used to produce the data in Table (3.3) is available on the author's website [Be]. For every exceptional group, the fake polynomials of the irreducible characters are listed there. The remainder of  $P(\mathbb{C}[\mathfrak{h}]^{coW}, t)$  on division by  $t^{-b^*} f_{\lambda^*}(t)$  is also listed. In addition, this information is available for many of the groups  $G(m, p, n)$  of rank  $\leq 5$ .

#### 4. THE EXCEPTIONAL GROUP $G_4$

The group  $G_4$ , as labelled in [ST], is the binary tetrahedral group. It can be realised as a finite subgroup of the group of units in the quaternions

$$G_4 = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$$

and has order 24. It is generated by the elements  $s_1 = \frac{1}{2}(-1 + i + j - k)$  and  $s_2 = \frac{1}{2}(-1 + i - j + k)$  and has presentation  $G_4 = \langle s_1, s_2 | s_1^3 = s_2^3 = (s_1 s_2)^6 = 1 \rangle$ . It has seven conjugacy classes which we label  $Cl_1 = \{1\}$ ,  $Cl_2$ ,  $Cl_3$ ,  $Cl_4$ ,  $Cl_5$ ,  $Cl_6$ , and  $Cl_7$ . The character table is

Class	1	2	3	4	5	6	7
Size	1	1	4	4	6	4	4
Order	1	1	3	3	4	6	6
$T$	1	1	1	1	1	1	1
$V_1$	1	1	$\omega^2$	$\omega$	1	$\omega^2$	$\omega$
$V_2$	1	1	$\omega$	$\omega^2$	1	$\omega$	$\omega^2$
$W$	2	-2	-1	-1	0	1	1
$\mathfrak{h}$	2	-2	$-\omega^2$	$-\omega$	0	$\omega^2$	$\omega$
$\mathfrak{h}^*$	2	-2	$-\omega$	$-\omega^2$	0	$\omega$	$\omega^2$
$U$	3	3	0	0	-1	0	0

where  $\omega$  is a primitive cube root of unity. Note that the reflection representation  $\mathfrak{h}$  has dimension 2, therefore  $G_4$  is a rank 2 complex reflection group.

The group  $G_4$  has two classes which consist of complex reflections and we label these reflections as

$$Cl_3 = \{s_1, s_2, s_3, s_4\}$$

$$= \{\frac{1}{2}(-1 + i + j - k), \frac{1}{2}(-1 + i - j + k), \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 - i - j - k)\}$$

and

$$\begin{aligned} Cl_4 &= \{t_1, t_2, t_3, t_4\} \\ &= \left\{ \frac{1}{2}(-1 - i - j + k), \frac{1}{2}(-1 + i - j - k), \frac{1}{2}(-1 - i + j - k), \frac{1}{2}(-1 + i + j + k) \right\} \end{aligned}$$

Unlike all other exceptional irreducible complex reflection groups we have

**Theorem 4.1.** *For generic values of  $\mathbf{c}$ , the generalised Calogero-Moser space  $X_{\mathbf{c}}$  associated to  $G_4$  is a smooth variety.*

*Proof.* The theorem is proved by showing that each irreducible  $H_{0,\mathbf{c}}$ -module is isomorphic to the regular representation of  $G_4$ . By [EG, Proposition 3.8], this is equivalent to the statement of the theorem. Let  $E = T \oplus V_1 \oplus V_2 \oplus 3U$  and  $F = \mathfrak{h} \oplus \mathfrak{h}^* \oplus W$ , two  $G_4$ -modules.

### Claim 1

Let  $L$  be a finite dimensional  $H_{0,\mathbf{c}}$ -module, then  $L \cong aE \oplus bF$ , for some  $a, b \in \mathbb{Z}_{\geq 0}$ .

To prove Claim 1 we use an argument similar to that of [EG, Proposition 16.5]. Let  $\rho : H_{0,\mathbf{c}} \rightarrow \text{End}_{\mathbb{C}}(L)$  realise the action of  $H_{0,\mathbf{c}}$  on  $L$ . Then, for all  $x, y \in \mathfrak{h} \oplus \mathfrak{h}^*$ , we have the commutation relation

$$(8) \quad [\rho(x), \rho(y)] = c_1 \sum_{i=1}^4 \omega_{s_i}(x, y) \rho(s_i) + c_2 \sum_{j=1}^4 \omega_{t_j}(x, y) \rho(t_j)$$

By Lemma 2.1,  $\sum_{i=1}^4 \omega_{s_i} = \sum_{j=1}^4 \omega_{t_j} = 2\omega$ . Taking traces on both sides of equation (8)

$$(9) \quad 0 = c_1 2\omega(x, y) \text{Tr}_L(s_1) + c_2 2\omega(x, y) \text{Tr}_L(t_1) \quad \forall x, y \in \mathfrak{h} \oplus \mathfrak{h}^*$$

Since  $c_1$  and  $c_2$  are generic i.e. take values in a dense open subset of  $\mathbb{C}^2$ , and equation (9) is linear, we have  $0 = 2\omega(x, y) \text{Tr}_L(s_1) = 2\omega(x, y) \text{Tr}_L(t_1)$ . The fact that  $\omega$  is nondegenerate implies that  $\text{Tr}_L$  is zero on  $Cl_3$  and  $Cl_4$ .

Using the fact that  $s_1$  is a complex reflection and  $\dim \mathfrak{h}^* = 2$ , we can choose a nonzero  $x_1 \in \mathfrak{h}^*$  such that  $s_1(x_1) = x_1$ . Then  $s_1[x_1, y] = [x_1, s_1(y)]$  for all  $y \in \mathfrak{h}$ . Since  $s_1(x_1) = x_1$ ,  $x_1 \in \text{Ker}(1 - s_1)$  and hence  $\omega_{s_1}(x_1, y) = 0$  for all  $y \in \mathfrak{h}$ . Similarly,  $s_1 t_1 = 1$  implies that  $x_1 \in \text{Fix}(t_1)$  and hence  $\omega_{t_1}(x_1, y) = 0$ . Therefore, multiplying both sides of equation (8) on the left by  $\rho(s_1)$  and taking traces

$$0 = c_1 \sum_{i=2}^4 \omega_{s_i}(x_1, y) \text{Tr}_L(s_1 s_i) + c_2 \sum_{j=2}^4 \omega_{t_j}(x_1, y) \text{Tr}_L(s_1 t_j)$$

Again, using the fact that  $c_1, c_2$  are generic, we get

$$0 = \sum_{i=2}^4 \omega_{s_i}(x_1, y) \text{Tr}_L(s_1 s_i) = \sum_{j=2}^4 \omega_{t_j}(x_1, y) \text{Tr}_L(s_1 t_j)$$

Since  $s_1s_2, s_1s_3$  and  $s_1s_4$  all belong to  $Cl_7$  and  $s_1t_2, s_1t_3, s_1t_4$  all belong to  $Cl_5$  we have

$$\begin{aligned} 0 &= \sum_{i=2}^4 \omega_{s_i}(x_1, y) Tr_L(s_1s_i) = 2\omega(x_1, y) Tr_L(s_1s_2) \\ 0 &= \sum_{j=2}^4 \omega_{t_j}(x_1, y) Tr_L(s_1t_j) = 2\omega(x_1, y) Tr_L(s_1t_2) \end{aligned}$$

Therefore  $Tr_L$  is zero on  $Cl_7$  and  $Cl_5$ .

We can also multiply both sides of equation (8) on the left by  $\rho(t_1)$  instead of  $\rho(s_1)$ . Noting that  $t_1^2 \in Cl_3$ ,  $t_1t_2, t_1t_3, t_1t_4 \in Cl_6$  and repeating the above argument shows that  $Tr_L$  is also zero on  $Cl_6$ .

Therefore any element of  $G_4$  that has nonzero trace on  $L$  must belong to  $Cl_1$  or  $Cl_2$ . Hence the character associated to  $L$  must take values  $(n, m, 0, 0, 0, 0, 0)$ , for some  $n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}$ , on the classes  $Cl_1, Cl_2, \dots, Cl_7$ . Taking inner products shows that

$$L \cong \frac{1}{|G_4|}(n+m)E \oplus \frac{2}{|G_4|}(n-m)F$$

Setting  $a = \frac{1}{|G_4|}(n+m)$  and  $b = \frac{2}{|G_4|}(n-m)$  proves Claim 1.

### Claim 2

Let  $L$  be an irreducible representation of  $H_{0,\mathbf{c}}$ , with  $\mathbf{c}$  generic. Then  $L$  must be isomorphic to  $E \oplus F$  or  $\mathbb{C}G_4$  as a  $G_4$ -module.

If  $L$  is irreducible then  $\dim L \leq 24$ . Therefore Claim 1 implies that  $L \cong E, 2E, nF, 1 \leq n \leq 4, E \oplus F$  or  $\mathbb{C}G_4$ . Assume that  $L$  is isomorphic to  $E$  as a  $G_4$ -module. The action of  $\mathfrak{h}^*$  on  $L$  defines a linear map  $\phi : \mathfrak{h}^* \rightarrow \text{End}_{\mathbb{C}}(E)$ . For  $w \in G_4$  and  $x \in \mathfrak{h}^*$ ,  $w x w^{-1} = {}^w x$  in  $H_{0,\mathbf{c}}$ . Therefore  $\phi({}^w x)(e) = {}^w x.e = w x w^{-1}.e = w(x.(w^{-1}e)) = w(\phi(x)(w^{-1}e))$ . The action of  $w \in G_4$  on  $f \in \text{End}_{\mathbb{C}}(E)$  is defined by  $(wf)(e) = w(f(w^{-1}e))$ . Therefore the map  $\phi : \mathfrak{h}^* \rightarrow \text{End}_{\mathbb{C}}(E)$  is  $G_4$ -equivariant. The  $G_4$ -module  $\text{End}_{\mathbb{C}}(E)$  decomposes as

$$\begin{aligned} \text{End}_{\mathbb{C}}(E) &\cong (T \otimes T) \oplus 2(T \otimes V_1) \oplus 2(T \otimes V_2) \oplus 6(T \otimes U) \oplus (V_1 \otimes V_1) \oplus 2(V_1 \otimes V_2) \oplus \\ &6(V_1 \otimes U) \oplus (V_2 \otimes V_2) \oplus 6(V_2 \otimes U) \oplus 9(U \otimes U) \cong 12T \oplus 12V_1 \oplus 12V_2 \oplus 36U \end{aligned}$$

This shows that  $\mathfrak{h}^*$  is not a summand of  $\text{End}_{\mathbb{C}}(E)$ . Therefore  $\phi$  must be the zero map. Similarly, the action of  $\mathfrak{h}$  must also be zero on  $E$ . Therefore the right hand side of equation (8) must also act as zero on  $E$ . In particular, it must act as zero on  $T \subset E$ . This means that

$$0 = c_1 \sum_{i=1}^4 \omega_{s_i}(x, y) + c_2 \sum_{j=1}^4 \omega_{t_j}(x, y) = 2(c_1 + c_2)\omega(x, y)$$

This is a contradiction because  $c_1, c_2$  are generic and  $\omega$  is nondegenerate. Hence  $L$  cannot be isomorphic to  $E$ . Repeating the above argument for  $F$  we have

$$\begin{aligned} \text{End}_{\mathbb{C}}(F) &\cong (\mathfrak{h} \otimes \mathfrak{h}) \oplus 2(\mathfrak{h} \otimes \mathfrak{h}^*) \oplus 2(\mathfrak{h} \otimes W) \oplus \\ &(\mathfrak{h}^* \otimes \mathfrak{h}^*) \oplus 2(\mathfrak{h}^* \otimes W) \oplus (W \otimes W) \cong 3T \oplus 3V_1 \oplus 3V_2 \oplus 9U \end{aligned}$$

Therefore  $\mathfrak{h}^*$  and  $\mathfrak{h}$  must act as zero on  $F$ . If we consider the right hand side of equation (8), this time restricted to  $W \subset F$  then we have

$$0 = c_1 \sum_{i=1}^4 \omega_{s_i}(x, y) \rho|_W(s_i) + c_2 \sum_{j=1}^4 \omega_{t_j}(x, y) \rho|_W(t_j)$$

Taking the trace of this equation gives  $0 = -2(c_1 + c_2)\omega(x, y)$ , which is a contradiction because  $c_1, c_2$  are generic and  $\omega$  is nondegenerate. Therefore  $L \not\cong F$ . The same reasoning shows that  $L$  cannot be isomorphic to  $2E$  or  $nF$ ,  $2 \leq n \leq 4$  either. This proves Claim 2.

### Claim 3

Let  $L$  be an irreducible  $H_{0,\mathbf{c}}$ -module. Then  $L$  cannot be isomorphic to  $E \oplus F$  as a  $G_4$ -module.

By Lemma 2.3, there exists a generalised Verma module  $\Delta_{\mathbf{c}}(M; p, q)$  and a surjective homomorphism  $\phi : \Delta_{\mathbf{c}}(M; p, q) \rightarrow L$ . As a  $G_4$ -module we have

$$\Delta_{\mathbf{c}}(M; p, q) = H_{0,\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*] \rtimes W_q} M \cong \mathbb{C}G_4 \otimes \text{Ind}_{(G_4)_q}^{G_4} M \cong k\mathbb{C}G_4$$

where  $(G_4)_q$  is the stabiliser of  $q \in \mathfrak{h}^*$  and  $k = [G_4 : (G_4)_q] \dim M$ . The generalised Verma module  $\Delta_{\mathbf{c}}(M; p, q)$  has a finite composition series. Each factor of this series must have dimension  $\leq 24$ . Therefore, by Claim 2, each factor is isomorphic to either  $\mathbb{C}G_4$  or  $E \oplus F$  as a  $G_4$ -module. Hence there exist  $m, n \in \mathbb{N}$  such that  $k\mathbb{C}G_4 \cong m\mathbb{C}G_4 \oplus n(E \oplus F)$  with  $n \geq 1$ . But then  $n(E \oplus F) \cong (k - m)\mathbb{C}G_4$ , which is a contradiction. This completes the proof of Claim 3 and the theorem.  $\square$

**Corollary 4.2.** *Let  $X$  be the symplectic singularity  $\mathfrak{h} \times \mathfrak{h}^*/G_4$ . There exists a symplectic resolution  $\pi : Z \rightarrow X$  of  $X$ .*

First we recall some definitions from [Ko, page 236], the reader should consult that article for details. A variety will mean a quasi-projective variety over  $\mathbb{C}$ . Let  $X, Y$  be normal varieties with  $K_X$   $\mathbb{Q}$ -Cartier and  $f : Y \rightarrow X$  a birational morphism. We can write

$$K_Y \equiv f^*(K_X) + A$$

If  $E$  is a prime exceptional divisor on  $Y$  then the *discrepancy* of  $E$  with respect to  $X$  (denoted  $a(E, X)$ ) is defined to be the coefficient of  $E$  in  $A$ . If  $f' : Y' \rightarrow X$  is another birational morphism and  $E' \subset Y'$  the birational transform of  $E$  on  $Y'$  then  $a(E, X) = a(E', X)$ . Therefore  $a(E, X)$  depends only on  $E$  and not on  $Y$ . The variety  $X$  is called *canonical* if  $a(E, X) \geq 0$  for all  $E$ .

*Proof.* The affine variety  $X$  is four dimensional and normal. By [Wa, Watanabe's Theorem]  $X$  has Gorenstein singularities and hence the canonical divisor  $K_X$  is trivial (and hence Cartier). The affine

variety  $V = \mathfrak{h} \times \mathfrak{h}^*$  is smooth and therefore  $V$  is canonical. Since  $G_4$  is a finite group, the quotient map  $\pi : V \rightarrow X$  is a finite dominant morphism and  $\pi^* K_X = \pi^* \mathcal{O}_X = \mathcal{O}_V = K_V$ . Therefore we can apply [Ko, Proposition 3.16] which says that  $X$  is canonical. Therefore the pair  $(X, \emptyset)$  is a Kawamata log terminal pair (as defined in [AHK]) and we can apply [AHK, Lemma 2.1] to conclude that there exists an effective  $\mathbb{Q}$ -factorial terminal pair  $(Y, B)$  together with a birational morphism  $f : Y \rightarrow X$  such that

$$K_Y + B \equiv f^*(K_X)$$

However as noted above we can write  $K_Y \equiv f^*(K_X) + A$  with  $A = -B$ . Since  $X$  is canonical  $a(E, X) \geq 0$  for all exceptional prime divisors  $E$  on  $Y$ . Hence  $A$  is an effective divisor. But  $B$  is also effective therefore  $A = B = 0$  and we deduce that  $f : Y \rightarrow X$  is a crepant morphism. As noted in [EG, Section 4.14],  $\{X_c\}_{c \in \mathbb{C}^2}$  is a Poisson deformation of  $X$ . Therefore Theorem 4.1 says that  $X$  has a smoothing by a Poisson deformation. Now we can apply Namikawa's result [Na, Theorem 2.4] and conclude that there exists a symplectic resolution  $\pi : Z \rightarrow X$ .  $\square$

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